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Gerard Cornuejols

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David Hartvigsen

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Carnegie-Mellon University

PITTSBURGH, PENNSYLVANIA 15213

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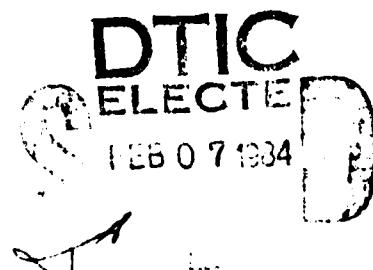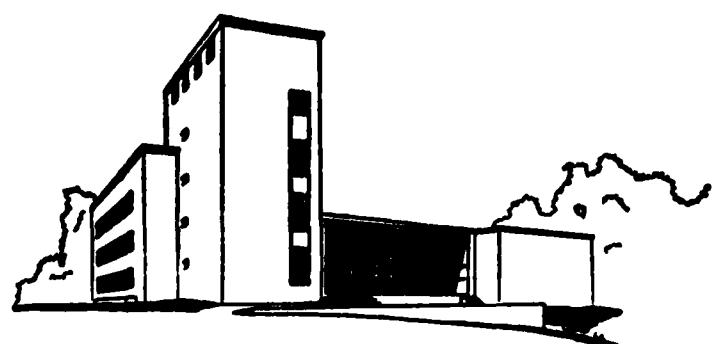
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Abstract

The authors introduces the notion of a hypomatching in a graph G as a collection of node disjoint edges and hypomatchable subgraphs of G where the hypomatchable subgraphs belong to some prespecified family. Examples include matchings, fractional matchings and edge-and-triangle packings. *It is* shown that many of the classical theorems about maximum cardinality matchings can be extended to hypomatchings which cover the maximum number of nodes in a graph.

1. Introduction

A <u>matching</u> of the nodes of a graph is a set of edges no two of which are adjacent. Some classical results about maximum cardinality matchings include the theorems of Gallai-Edmonds [5], Tutte [11], Urhy [12], Balas [1], Edmonds-Fulkerson [6], and Berge [2]. In this paper we extend the notion of a matching and show that these theorems still hold.

A <u>perfect matching</u> M is a matching such that every node of the graph is incident with one edge of M. A <u>near-perfect matching</u> is a matching which matches all but one of the nodes. A graph H is <u>hypomatchable</u> if its node set N has an odd cardinality and, for every $j \in N$, there is a near-perfect matching of H which leaves the node j unmatched. For example, odd cycles are hypomatchable. In fact, any hypomatchable graph H can be obtained as the graph H^p, $p \geq 0$, in a sequence of graphs $H^o \subseteq \ldots \subseteq H^p \equiv H$ such that H^o is a single node or an odd cycle; for $i \geq 1$, $H^{i+1} = H^i \cup P^i$ where (i) P^i is a path with both endnodes in H^i or a cycle with one node in H^i, (ii) no other node of P^i belongs to H^i and (iii) P^i has an odd number of edges. The sequence $H^o, \ldots, H^p$ was introduced by Lovasz [8] and is called an <u>ear decomposition</u> of H.

Given a graph G and a subset S of its nodes, G(S) denotes the subgraph of G induced by the node set S. We will say that the node set S is hypomatchable if G(S) is hypomatchable.

Consider a graph G and a family F of subsets of nodes of G. An <u>F-packing</u> of G is a disjoint subfamily $J \subseteq F$, i.e., every node of G belongs to at most one member of J. In this paper we will identify an edge and the set consisting of the two endnodes of that edge. Let E(G) be the edge set of G. When $F \equiv E(G)$, an E(G)-packing is simply a matching.

When H denotes a family of hypomatchable node sets and $F=E(G) \cup H$, F-packings will be called <u>hypomatchings</u>. Clearly, every matching of G is a hypomatching. Given a hypomatching J, any node which belongs to one member of J is said to be <u>covered</u> by J. A <u>maximum hypomatching</u> is one which covers the maximum number of nodes of G. When $H=\emptyset$ the (maximum) hypomatchings of G are precisely the (maximum) matchings of G. Apart from the matching problem, several examples of maximum hypomatching problems have appeared in the literature:

(i) Packing edges and triangles in a graph [4]: $S\epsilon H$ if and only if G(S) is a triangle (i.e., a complete graph on 3 nodes.)

(ii) Clique-packing [4], [7]: H is a family of odd cliques (i.e., node sets all of whose members are pairwise adjacent.)

(iii) P_k-matchings [3]: $S\epsilon H$ if and only if it is the node set of an odd cycle with more than k edges.

(iv) Dynamic matchings [10]: Weights are associated with the edges of an auxiliary graph G and $S\epsilon H$ if and only if S is the node set of an odd cycle of G for which the sum of the edge weights is zero.

(v) Fractional matchings [12], [1], [9] also known as 2-matchings: $S\epsilon H$ if and only if it is the node set of some odd cycle of G.

In [4], an algorithm was given for finding a maximum hypomatching in a graph. Here we will show that much of matching theory generalizes to hypomatchings. Our treatment does not assume the algorithm of [4]; instead it uses relationships between maximum matchings and maximum hypomatchings.

A <u>perfect hypomatching</u> of a subgraph G(S) is a hypomatching of G which covers all the nodes of S but no other node. A useful concept for

the theory of hypomatchings is that of a critical graph. The graph G(S) is __critical__ relative to F if it is hypomatchable and does not have a perfect hypomatching. The following theorem is proved in [4].

__Theorem 1__ A graph G(S) is critical if and only if it does not have a perfect hypomatching but, for every $j \in S$, the graph G(S-{j}) has one.

Theorem 1 justifies the terminology "critical graph" but will not be used explicitly in this paper.

2. __A Relationship Between Maximum Matchings and Maximum Hypomatchings__

Given a graph G, consider the following partition of its nodes into three sets O, I, R.

(i) a node of G belongs to O if and only if it is not matched in at least one maximum matching;

(ii) I is the set of nodes of G which are matched in every maximum matching and are adjacent to at least one node of O;

(iii) R is the set of nodes of G which are matched in every maximum matching but are not adjacent to any node of O.

The Gallai-Edmonds theorem states that

(a) every component of G(O) is hypomatchable;

(b) a matching of G is a maximum matching if and only if

(iv) the nodes of R are matched among themselves;

(v) in each component of G(O), all but one of the nodes are matched among themselves;

(vi) each node of I is matched to a node in a distinct component of G(O).

The partition O, I, R can be obtained by applying Edmonds' matching algorithm [5]. Consider the alternating forest at termination of the algorithm. The set of nodes which are either outer nodes of the forest or

inside shrunk outer nodes is the set O. The set of inner nodes of the forest forms the set I. The rest of the nodes of G is R. (The letters O, I, and R stand for outer, inner, and remaining nodes respectively.)

Now we turn to maximum hypomatchings. Recall that $F = E(G) \cup H$ where every $S \in H$ is a hypomatchable subset of the nodes of G. Consider the following partition of the nodes of G into three sets O(F), I(F), and R(F).

(i') a node of G belongs to O(F) if and only if it is not covered in at least one maximum hypomatching;

(ii') I(F) is the set of nodes of G which are covered in every maximum hypomatching and are adjacent to at least one node of O(F);

(iii') R(F) is the set of nodes of G which are covered in every maximum hypomatching but are not adjacent to any node of O(F).

Given a hypomatching J of $G = (V,E)$, a node of $S \subseteq V$ is said to be __internally covered__ in S if it is covered by a member T of J such that $T \subseteq S$.

__Theorem 2__ The partition O(F), I(F), R(F) is such that

(a') every component of G(O(F)) is critical;

(b') a hypomatching of G is maximum if and only if

(iv') all the nodes of R(F) are internally covered in R(F);

(v') in each component of G(O(F)), all but one of the nodes are internally covered in the component;

(vi') each node of I(F) is matched to a node in a distinct component of G(O(F)).

In order to prove Theorem 2 we need the following lemma.

__Lemma 3__ Let S and T be two subsets of the nodes of $G = (V,E)$ and assume that T is hypomatchable. If S and T have p (≥ 1) common nodes, then at most p-1 critical connected components of G(V-S) have a node set C such that G(C-T) admits a perfect hypomatching.

Proof of Lemma 3: Let $\hat{M}$ be a near perfect matching of G(T) leaving one node of S unmatched. Now let C be the node set of any critical connected component of G(V-S) such that the nodes of $C \cap T$ are matched among themselves by the matching $\hat{M}$. If G(C-T) had a perfect hypomatching, then completing it with the edges of $\hat{M}$ in $G(C \cap T)$ would produce a perfect hypomatching of G(C), a contradiction to the fact that G(C) is critical. So the critical components of G(V-S) such that G(C-T) has a perfect matching must have at least one of their nodes matched with a node of S in the matching $\hat{M}$. There are at most p-1 such components since $\hat{M}$ leaves one node of S unmatched.

Now we prove the theorem.

Proof of Theorem 2: Consider the sets 0, I and R defined by (i) - (iii). Let $\bar{G}$ be the bipartite graph obtained from $G(0 \cup I)$ by shrinking each connected component of G(0) to a single node and by removing all the edges of G(I). If a component of G(0) is critical, the corresponding node of $\bar{G}$ will also be called critical. As a consequence of statement (vi) of the Gallai-Edmonds theorem, every maximum matching of $\bar{G}$ matches all the nodes of I. Let $\bar{M}$ be such a maximum matching of $\bar{G}$ with the property that the number of critical matched nodes is the largest possible among all maximum matchings of $\bar{G}$.

If every critical node of $\bar{G}$ is matched by $\bar{M}$, set R(F) to be the node set of G and O(F) = I(F) = ∅. Otherwise, let the critical unmatched nodes of $\bar{G}$ be defined as the roots of the trees of a forest A. These nodes will also be called outer nodes of A. If some edge e joins an outer node of A to a node i∈I not in A, let m=(i,j) be the edge of $\bar{M}$ incident with i. Grow the forest A by adding to A the edges e and m and call the nodes i and j inner and outer nodes of A respectively. (Note that the node j must

be critical, otherwise by interchanging the edges in and out of $\bar{M}$ on the path of A from j to the root, one more critical node could be matched, contradicting the assumption about $\bar{M}$.) Keep growing the forest A as described above until every edge incident with an outer node of A is also incident with an inner node of A.

Then let I(F) be the set of inner nodes of A, O(F) the set of nodes of G(0) contained in outer nodes of A, and R(F) the remaining nodes of G. So $I(F) \subseteq I$, $O(F) \subseteq O$ and $R(F) \supseteq R$. Note also that, by construction of A, every component of G(O(F)) is critical and no edge of G joins O(F) to R(F). We will show that the partition O(F), I(F), R(F) just constructed is in fact the unique partition defined by (i') – (iii').

Before doing this, we exhibit a hypomatching J of G which leaves s uncovered nodes, where s is defined to be the number of components of G(O(F)) minus the cardinality of I(F). We define J separately on G(R), $\bar{G}$ and G(0). In G(R), take J to be any perfect matching (this is possible by statement (iii) of the Gallai-Edmonds theorem). In $\bar{G}$, take J to be identical to $\bar{M}$. Finally, in G(0), take J to be a hypomatching which internally covers all the nodes of the noncritical components incident with no edge of $\bar{M}$, and all but one of the nodes of the remaining components of G(0). (When such a component contains a node u incident with $\bar{M}$, u is the only node of the component which is not internally covered). Since $\bar{M}$ matches every node of I(F) with a node of O(F), so does J, leaving only s uncovered nodes in O(F). Every node of R(F) is covered by J, so we have the announced hypomatching.

In fact, the hypomatching J just constructed is maximum:

A consequence of Lemma 3 with $S \equiv I(F)$ is that any hypomatching of G which does not cover all the nodes of I(F) or which contains a

hypomatchable set $T \in H$ with at least one node in $I(F)$, must leave more than s uncovered nodes in $O(F)$. By matching all the nodes of $I(F)$ to nodes of $O(F)$ at least s nodes of $O(F)$ must remain uncovered, and in fact it is possible to leave exactly s uncovered nodes in G, as shown by the hypomatching J constructed earlier. This shows that J is a maximum hypomatching.

This also proves that every maximum hypomatching of G satisfies (iv'), (v') and (vi'). Conversely any hypomatching which satisfies (iv'), (v') and (vi') leaves only s uncovered nodes and therefore is maximum.

The fact that maximum hypomatchings satisfy (iv') and (vi') implies (ii') and (iii'). So only statement (i') remains to be proved. Consider the forest A in $\tilde{G}$. Any critical outer node j of A can be left unmatched by some matching $\tilde{M}$ which has the same cardinality as $\bar{M}$ and leaves unmatched the same noncritical nodes as $\bar{M}$. Specifically, if j is a critical node of A matched by $\bar{M}$, then construct $\tilde{M}$ from $\bar{M}$ by interchanging the edges in and out of $\bar{M}$ on the path of A from j to a root of A. Now the matching $\tilde{M}$ can be used instead of $\bar{M}$ to construct a maximum hypomatching J as done earlier. Furthermore, in the critical component of $G(O(F))$ left unmatched by $\tilde{M}$, any node can be left uncovered. This proves statement (i') and completes the proof of Theorem 2.

This structural theorem has many consequences, as we shall see in the next four theorems.

<u>Theorem 4</u> Consider a graph G and two families $F_1 = E(G) \cup H_1$ and $F_2 = E(G) \cup H_2$ such that the node sets in H_1 and H_2 are hypomatchable. If $H_1 \subseteq H_2$, then the partitions $O(F_i)$, $I(F_i)$, $R(F_i)$, $i=1,2$, satisfy $O(F_2) \subseteq O(F_1)$, $I(F_2) \subseteq I(F_1)$ and therefore $R(F_2) \supseteq R(F_1)$].

Proof: The property $O(F_2) \subseteq O(F_1)$ follows from (i') and the fact that every hypomatching relative to F_2 is also a hypomatching relative to F_1.

The property $I(F_2) \subseteq I(F_1)$ follows from $O(F_2) \subseteq O(F_1)$ and the fact that $I(F_i)$ is exactly the set of nodes of G adjacent to $O(F_i)$, i=1,2 (see (ii') and (iii')).

The next result generalizes a theorem of Tutte [11].

Theorem 5 A graph $G=(V,E)$ has a perfect hypomatching if and only if, for every $S \subseteq V$, the graph $G(V-S)$ contains at most $|S|$ critical connected components.

Proof: If G does not have a perfect hypomatching, then $O(F) \neq \emptyset$ in Theorem 2. Let $S = I(F)$. By Theorem 2, the number of critical components in $G(O(F))$ is larger than $|S|$ (since a maximum hypomatching matches each node of S with a node in a different component of $G(O(F))$ and still leaves at least one component of $G(O(F))$ unmatched). By (iii') the critical components in $G(O(F))$ remain critical in $G(O(F) \cup R(F)) \equiv G(V-S)$.

Conversely, assume that G has a perfect hypomatching J. Consider any $S \subseteq V$. If no hypomatchable set $T \in J$ contains a node of S, then every critical component of $G(V-S)$ has to be matched to some node of S by an edge of J, proving the theorem. Otherwise, $|(T \cap S)| = p \geq 1$ for some $T \in J$. Then by Lemma 3, the number of critical components of $G(V-S)$ having a node set C such that $G(C-T)$ admits a perfect hypomatching is at most p-1. The theorem follows by induction on the number of hypomatchable sets of J which intersect S.

3. Maximum Hypomatchings with a Minimum Number of Hypomatchable Sets

The next result generalizes a theorem of Urhy relating maximum matchings and fractional matchings [12]. Let $G = (V,E)$ be a graph and let $F = E \cup H$ where H is a family of hypomatchable sets.

Theorem 6 Let $J \subseteq F$ be a maximum hypomatching containing a minimum number of hypomatchable sets. Then the matching obtained by taking the edges of J and near perfectly matching the hypomatchable sets of J, is a maximum matching.

Our proof of Theorem 6 uses the following lemma.

Lemma 7 A noncritical hypomatchable subgraph K of G has a perfect hypomatching using only one of the hypomatchable sets in H.

A proof of this lemma can be found in [4]; it is based on a simple alternating path argument to reduce the number of hypomatchable sets in any perfect hypomatching of K containing more than one such set.

Proof of Theorem 6: Let 0, I, R be the node sets defined in the Gallai-Edmonds theorem and $O(F)$, $I(F)$, $R(F)$ those defined in Theorem 2. Set $I-I(F) = L$ and $0-0(F) = \Omega$.

Consider J as defined in Theorem 6. By Theorem 2 every node of $I(F)$ is matched by an edge of J to a node of $0(F)$ and in every connected component of $G(0(F))$ all but one of the nodes are internally covered. Only edges are needed in these near perfect hypomatchings since the components of $G(0(F))$ are hypomatchable. So the nodes of $0(F) \cup I(F)$ are only matched by edges of J.

The nodes of $R(F) = R \cup \Omega \cup L$ are internally covered. Since the nodes of Ω are only joined to L in $G(R(F))$, the number of hypomatchable subgraphs needed to internally cover the nodes of $R(F)$ is at least equal to the number of components of Ω minus the cardinality of L. In fact, the matching $\bar{M}$ defined in the proof of Theorem 2 shows that no more are needed since (1) the nodes of R are perfectly matched among themselves, (2) every node of L is matched to a component of Ω and (3) in each component of Ω which is not matched to L, all the nodes can be internally covered using

only one hypomatchable set by Lemma 7.   This completes the proof of Theorem 6.

A set of nodes S is <u>separable</u> if and only if there exists a maximum matching which does not use any edge with exactly one end in S.   The next result generalizes a theorem of Balas [1].

<u>Theorem 8</u>   Given $G = (V,E)$ and $F = E \cup H$, a maximum matching is also a

maximum hypomatching of G if and only if none of the hypomatchable

sets in H is separable.

<u>Proof</u>:   The necessity follows from the observation that, if some hypomatchable set $S \in H$ were separable, then a maximum matching M using no edge in the boundary of S would leave one node of S unmatched, but a hypomatching identical to M on $G(V-S)$ and using S would cover one more node of G.

Conversely, suppose G does not have a maximum hypomatching using just edges.   Consider one which uses a minimum number of hypomatchable sets of H.   By Theorem 6, these sets are separable.   This completes the proof.

4.   <u>Hypomatching Matroid</u>

Let $G = (V,E)$ and $F = E \cup H$ where H is a family of hypomatchable sets. A node set $S \subseteq V$ is said to be <u>independent</u> if there exists a hypomatching $J \subseteq F$, such that S is a subset of the nodes covered by J.   Let M be the family of all independent sets.   The system (V,M) is an independence system, i.e., $S \in M$ and $T \subseteq S \Rightarrow T \in M$.   When a hypomatching covers all the nodes of a set S, we say that it covers S.

<u>Theorem 9</u>   The independence system (V,M) is a matroid.

<u>Proof</u> = Consider $G(O(F) \cup I(F))$.   In this graph, we say that a node set $X \in M'$ if and only if X can be covered by a matching of $G(O(F) \cup I(F))$.   It

is known (Edmonds Fulkerson [6]) that the independence system $(O(F) \cup I(F),$ M') so defined is a matroid, the so called matching matroid.

Note the following relationship between M and M'. $S \epsilon M$ if and only if $S = X \cup Y$ where $X \epsilon M'$ and $Y \subseteq R(F)$, as a consequence of Theorem 2. Since (V,M) is the direct sum of a matching matroid, namely $(O(F) \cup I(F),$ M') and a complete matroid, namely $(R(F), 2^{R(F)})$, it is itself a matroid.

Let $\omega: V \rightarrow R^+$ be a vector of nonnegative weights defined on the node set of G. The <u>weight</u> of a hypomatching J is defined as $\omega(J) = \Sigma \{\omega_i :$ node i is covered by J}. Consider the problem of finding a maximum weight hypomatching in G. Since (V,M) is a matroid, a maximum weight hypomatching can be found by the following <u>greedy algorithm</u>.

Order the nodes by nonincreasing weights $\omega_1 \geq \dots \geq \omega_n$. Start with $S^o = \emptyset$ and $J^o = \emptyset$. Then n iterations are performed, say iterations i = 1,...,n. At the beginning of iteration i, the set S^{i-1} is a maximum weight independent subset of $\{1,\dots,i-1\}$ and J^{i-1} is a hypomatching covering S^{i-1}. Iteration i consists of either proving that $S^{i-1} \cup \{i\}$ is not independent, or setting $S^i = S^{i-1} \cup \{i\}$ and modifying J^{i-1} (if necessary) into a hypomatching J^i which covers the set S^i. The algorithm terminates when i = n. The hypomatching J^n is a maximum weight hypomatching in G. Its weight is

$$\omega(J^n) = \underset{i \epsilon S^n}{\Sigma} \omega_i.$$

Next, we show how iteration i of this greedy algorithm can be performed by a variation of the Edmonds matching algorithm. More generally, let S be an independent set in (V,M) and J a hypomatching covering S. The next algorithm will check whether $S \cup \{i\}$ is independent, where $i \epsilon V-S$ is given, and if so, modify J so that it covers $S \cup \{i\}$.

First, if i is covered by J we can stop immediately and conclude that S ∪{i} is independent. Otherwise, we will construct a tree A with root i in an associated graph $\bar{G}$. Initially $\bar{G}$ = G and the root i is the unique node of A and it is said to be an even node. Then A is grown according to the following procedure until either S ∪{i} is found to be independent or A cannot be grown any longer in which case we will show that S ∪{i} is not independent.

<u>Step 1</u> If every edge of $\bar{G}$ which is incident with an even node of A is also incident with an odd node of A, stop: The set S ∪ {i} is not independent (this claim will be proved later). Otherwise, let j be an edge which joins an even node of A, say u, to a node v which is not an odd node of A. If v is an even node of A, go to Step 2. If v is not in A but is covered by an edge k = (vw) of J such that w∈S, then go to Step 3. Finally, in the other cases where v is not in A, go to Step 4.

<u>Step 2</u> Let $\bar{C}$ = {u,...,v} be the set of nodes in the unique path of the tree A joining nodes u and v, and let C be the set of nodes of G associated with the node set $\bar{C}$. C is hypomatchable.

If C ⊆ S and G(C) is critical, modify A (and $\bar{G}$) by shrinking $\bar{C}$ to a single node. This shrunk node becomes an even node of A. Go to Step 1.

Otherwise, modify J by alternating the edges in and out of J on the path of A from i to the closest point in $\bar{C}$. If necessary, modify the near perfect matchings inside the shrunk even nodes on this path so that every node of G is in at most one member of J. (This is always possible since the shrunk nodes of A are hypomatchable.) In addition, if there exists w∈C−S, J is modified in G(C) so as to contain a near perfect matching of G(C) leaving w uncovered; on the other hand, if C ⊆ S, then G(C) is not critical and J is modified in G(C) so as to internally cover the nodes of

G(C). This produces a hypomatching J' which covers $S \cup \{i\}$. Stop.

<u>Step 3</u> Grow the tree A by adding the edges j and k and the nodes v and w
to A. Node v is called an odd node of A and w an even node. Go to Step 1.

<u>Step 4</u> The node v is not in A and is either (i) not covered by J, or (ii)
covered by an edge (vw) of J such that $w_\epsilon S$, or (iii) covered by a hypo-
matchable set of J. Let J' be obtained from J by interchanging in and out
of J the edges on the path of A joining i to v. If necessary, modify the
near-perfect matchings inside the shrunk even nodes on this path. In
addition, in case (ii), remove the edge (vw); in case (iii) replace the
hypomatchable set T of J which covers v by a near-perfect matching of T
leaving only v uncovered. Now, in all 3 cases, J' is a hypomatching which
covers $S \cup \{i\}$. So $S \cup \{i\}$ is independent. Stop.

<u>Proof of the Validity of the Algorithm</u>: It is clear that this algorithm
terminates since every time it goes back to Step 1 a new edge of G is
considered. When the algorithm terminates in Steps 2 or 4, the
hypomatching J' proves that $S \cup \{i\}$ is independent. So in order to prove
the validity of the algorithm it suffices to show that, when the algorithm
terminates in Step 1, the set $S \cup \{i\}$ is not independent. By construction
of A, the even nodes of A which are shrunk only contain nodes of S (Step
2) and the other even nodes of A belong to S(Step 3). Also by
construction the tree A contains one more even node than odd. Finally,
when the algorithm terminates every edge incident with an even node of A
has an odd node of A as its other endpoint. As a consequence of Lemma 3,
no hypomatching of G can cover all the nodes inside critical components of
G(V—I) where I is the set of odd nodes of A, since there are $|I|+1$ such
critical components. Thus, no hypomatching covers all the nodes of $S \cup$
$\{i\}$, proving that this set is not independent.

We conclude with a generalization of a theorem of Berge [2]. An alternating path relative to a hypomatching J is a path whose edges are alternately in and out of J. An augmenting path is an alternating path one of whose endpoints is not covered by J and whose other endpoint u is either

(a) not covered by J, or

(b) in a hypomatchable set of J, or

(c) in a noncritical hypomatchable graph G(C) such that the nodes of C-{u} are matched among themselves by J. In addition , the length of the alternating path must be even.

Note that in cases (a) and (b) the length of the alternating path will always be odd.

Theorem 10 A hypomatching is maximum if and only if there exists no augmenting path.

Proof: If (a), (b), or (c) occurs, the hypomatching J is not maximum. Coversely assume that J is not maximum. Let S be the set of nodes covered by J and let i be a node such that S $\cup$ {i} can be covered. By the algortihm we will find an augmentation. It occurs either in Step 2, providing an augmenting path as stated in (c), or in Step 4, providing augmenting paths (a) or (b). (Note that Step 4 case (ii) does not occur with our choice of S.)

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